$$\delta I^{S/S^*} = [Mm/(M+m)](2\delta r_P \cdot r_P U - \delta r_P r_P - r_P \delta r_P) \tag{9}$$

The order of magnitude of this change can be expressed more succinctly by letting R_G be the radius of gyration of B. Thus, we can express $\|\delta I^{S/S^*}\|$ as

$$\left\| \frac{\delta I^{S/S^*}}{I^{S/S^*}} \right\| \sim \mu_R \left(\frac{\delta r_P}{R_G} \right) \left(\frac{r_P}{R_G} \right) \tag{10}$$

which shows that the fractional change in the inertia is a second-order effect and may even be reduced to a third-order effect by keeping $r_P/R_G \sim \mathcal{O}(\varepsilon)$. Similar to the inertia dyadic, an expression for the central angular momentum may be written as

$$H^{S/S^*} = I^{S/S^*} \cdot {}^N\omega^B + I^{P/P^*} \cdot {}^B\omega^P + [mM/(M+m)] r_P \times {}^B V^{P^*} \tag{11}$$

where ${}^N\omega^B$ is the angular velocity of B with respect to N, a Newtonian frame, and ${}^B V^{P^*}$ is the velocity of P^* with respect to B. The last two terms of this equation may be viewed as extra terms in the original angular momentum. Because these are either precisely known quantities, for example, ${}^B\omega^P = \mathbf{0}$ for a sliding proof mass, or quantities whose effects can be calibrated, it can be taken into account without much difficulty in a precision attitude control system. Note also that for precision control, we may require ${}^B V^{P^*} \sim \mathcal{O}(\varepsilon)$ or perhaps even ${}^B V^{P^*} \sim \mathcal{O}(\varepsilon^n) n > 1$ so that the last term is Eq. (11) can be made quite small. In any case, for the precision orbital maneuvering system proposed in this Note, an attitude control system that accounts for the orbit-attitude coupling is necessary.

Conclusions

In any mechanism that purports precision control, there are the usual issues in precision manufacturing. Our mechanism is no different in this context. However, the two main features of our mechanism are 1) the possibility to achieve orbit control without the use of mass expulsion and 2) the ability to achieve second-order precision in the state vector from a first-order precision in control. One limitation of this mechanism is that it can be used only for periodic disturbances that generate perturbations that are no larger than the characteristic length of the spacecraft. For perturbations that are secular, the mechanism can still be used, but periodic reinitialization, similar to the practice of momentum dumping in attitude control, is necessary. Further work on how this mechanism couples with the attitude control system is warranted.

Acknowledgments

I would like to thank Tom Thorvaldsen and Matt Bottkol, both of Charles Stark Draper Laboratories, for encouraging me to pursue the ideas presented here.

References

[1]Bender, P., Brillet, A., Ciufolini, I., Cruise, A. M., Cutler, C., Danzmann, K., Fidecaro, F., Folkner, W. M., Hough, J., McNamara, P., Peterseim, M., Robertson, D., Rodrigues, M., Rüdiger, A., Sandford, M., Schäfer, G., Schilling, R., Schutz, B., Speake, C., Stebbins, R. T., Sumner, T., Touboul, P., Vinet, J.-Y., Vitale, S., Ward, H., and Winkler, W., "LISA: Laser Interferometer Space Antenna, Pre-Phase A Report," Max-Planck-Inst., Rept. MPQ 208, Garching, Germany, Dec. 1995.

[2]Bender, P., Ciufolini, I., Danzmann, K., Folkner, W. M., Hough, J., Robertson, D., Rodrigues, M., Rüdiger, A., Sandford, M., Schilling, R., Schutz, B., Stebbins, R. T., Sumner, T., Touboul, P., Vitale, S., Ward, H., and Winkler, W., "LISA: Laser Interferometer Space Antenna, Pre-Phase A Report," 2nd ed., Jet Propulsion Lab., JPL Rept. MPQ 233, California Inst. of Technology, Pasadena, CA, July 1998.

[3]*New World Vistas*, Summary Vol., U.S. Air Force Scientific Advisory Board, Dec. 1995.

[4]Carter, T. E., "State Transition Matrices for Terminal Rendezvous Studies: Brief Survey and New Examples," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 1, 1998, pp. 148–155.

[5]Melton, R. G., "Time-Explicit Representation of Relative Motion Between Elliptical Orbits," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 4, 2000, pp. 604–610.

[6]Vadali, S. R., Alfriend, K. T., and Vaddi, S. S., "A New Concept for Constellation Control of Formation Flying Satellites," American Astronautical Society, Paper AAS 01-218, Feb. 2001.

[7]Kaplan, M. H., *Modern Spacecraft Dynamics and Control*, Wiley, New York, 1976, p. 111.

[8]Kane, T. R., Likins, P. W., and Levinson, D. A., *Spacecraft Dynamics*, McGraw–Hill, New York, 1983, p. 213.

Consistent Approximations to Aircraft Longitudinal Modes

N. Ananthkrishnan* and P. Ramadevi†
Indian Institute of Technology Bombay, Mumbai 400 076, India

I. Introduction

EVER since Lanchester[1] derived an approximation to the phugoid frequency, literal approximations have formed an important part of the theory of aircraft dynamic stability.[2–4] However, the standard phugoid approximation often fails to match well with aircraft data, whereas the short period approximation is usually found to be numerically accurate. This has led to the derivation of several alternate phugoid approximations, many of which have been reviewed by Pradeep.[5] Unfortunately, the relative merits of the competing phugoid approximations could only be compared by matching them with numerical data, and none of them proved to be uniformly satisfactory. Part of this mystery has been resolved in a recent work,[6] which uncovered a serious flaw in previous derivations that led to incorrect approximations to the slow modes in general, and the phugoid mode in particular. Meanwhile, nearly a century after Lanchester, researchers are still engaged in deriving improved approximations to the phugoid mode.[5,7] Pradeep[5] appears to have been the first to realize that "correct" approximations to the longitudinal modes should satisfy the fourth-order longitudinal characteristic polynomial. Then, there would be no need to test this approximation for its correctness against numerical data, nor could this approximation be improved upon in any manner. Believing the existing short period approximation to be correct, Pradeep attempted to derive a correct phugoid approximation by factoring out the second-order short period polynomial from the fourth-order characteristic polynomial. However, his formulation had just two unknowns, which could not satisfy all four equations for the coefficients of the characteristic polynomial.

Rather than start with the characteristic polynomial, in the present Note we follow a formal procedure to derive systematically correct literal approximations to the aircraft longitudinal modes. No assumption is made except that there exist two distinct timescales corresponding to the fact that the short period mode is notably faster than the phugoid for most conventional airplanes. The ratio of the two timescales, which is small, is then exploited as an order parameter, and approximations are derived to zeroth- and first-order in the small parameter. These approximations are consistent in the sense that they retain all terms up to a certain order in the small parameter, thus ensuring that there are no missing terms in the approximations at each order. The consistent approximations are shown to match all of the coefficients of the characteristic polynomial perfectly at each order. The present derivation is the first instance where the literal approximations have been shown to check with all of the coefficients of the characteristic polynomial. In particular, the existing short period approximation is shown to be not consistent at the first order, which partly explains Pradeep's failure to derive a correct phugoid approximation by factorization.

Received 7 May 2001; revision received 1 March 2002; accepted for publication 7 March 2002.

*Assistant Professor, Department of Aerospace Engineering; akn@aero.iitb.ac.in. Senior Member AIAA.

†Assistant Professor, Department of Physics; ramadevi@phy.iitb.ac.in.

II. Linearized Equations

We begin with the second-order form of the linearized equations for perturbed longitudinal dynamics about a straight and level flight trim at velocity V_0, as follows[6]:

$$\ddot{\Delta}\alpha - (Z_u)\dot{\Delta}v - (M_q + Z_\alpha/V_0)\dot{\Delta}\alpha - (V_0 M_u - M_q Z_u)\Delta v - [M_\alpha - M_q(Z_\alpha/V_0)]\Delta\alpha = 0 \tag{1}$$

$$\ddot{\Delta}v - (X_u)\dot{\Delta}v - [(X_\alpha - g)/V_0]\dot{\Delta}\alpha - (gZ_u/V_0)\Delta v - \left(gZ_\alpha/V_0^2\right)\Delta\alpha = 0 \tag{2}$$

where $\Delta\alpha$ is the perturbed angle of attack and $\Delta v = \Delta V/V_0$ is the dimensionless perturbed velocity. The short period is represented by Eq. (1) for the angle of attack, whereas Eq. (2) for the velocity is associated with the phugoid mode. The two timescales in the longitudinal dynamics become apparent when the dimensional derivatives in Eqs. (1) and (2) are replaced with their standard definitions (for example, see Ref. 4, p. 6.17) to give

$$\ddot{\Delta}\alpha - \left[\left(\frac{g}{V_0}\right)z_u\right]\dot{\Delta}v - \left[\sqrt{\frac{\bar{q}Sc}{I_y}}m_q + \left(\frac{g}{V_0}\right)z_\alpha\right]\dot{\Delta}\alpha - \left[\frac{\bar{q}Sc}{I_y}m_u - \sqrt{\frac{\bar{q}Sc}{I_y}}\left(\frac{g}{V_0}\right)m_q z_u\right]\Delta v - \left[\frac{\bar{q}Sc}{I_y}m_\alpha - \sqrt{\frac{\bar{q}Sc}{I_y}}\left(\frac{g}{V_0}\right)m_q z_\alpha\right]\Delta\alpha = 0 \tag{3}$$

$$\ddot{\Delta}v - \left[\left(\frac{g}{V_0}\right)x_u\right]\dot{\Delta}v - \left[\left(\frac{g}{V_0}\right)(x_\alpha - 1)\right]\dot{\Delta}\alpha - \left[\left(\frac{g}{V_0}\right)^2 z_u\right]\Delta v - \left[\left(\frac{g}{V_0}\right)^2 z_\alpha\right]\Delta\alpha = 0 \tag{4}$$

where x, z, m with subscripts u, α, q are used as shorthand notation for bookkeeping the nondimensional terms associated with the dimensional stability derivatives. The precise expressions for these terms are not of interest here because they will be replaced with the corresponding dimensional derivatives when the final results are presented. It is obvious that there exist two distinct timescales in Eqs. (3) and (4), a fast timescale T_f of the short period motion and a slow timescale T_s for variation of the phugoid mode, with their ratio ϵ, therefore, appearing as a small parameter, as follows:

$$\frac{1}{T_f} = \sqrt{\frac{\bar{q}Sc}{I_y}}, \qquad \frac{1}{T_s} = \frac{g}{V_0}, \qquad \epsilon = \frac{T_f}{T_s} = \frac{g/V_0}{\sqrt{\bar{q}Sc/I_y}} \tag{5}$$

Using the timescales in Eq. (5), two nondimensional times t_f and t_s can be defined as follows:

$$t_f = t/T_f = t\sqrt{\bar{q}Sc/I_y}, \qquad t_s = t/T_s = t(g/V_0) \tag{6}$$

Before using the nondimensional times in Eq. (6) to rewrite Eqs. (3) and (4), it is necessary to distinguish between fast mode and slow mode variables. It was shown in Ref. 6 that there exists a component of $\Delta\alpha$, called the static residual, which varies as per the slow mode timescale. The concept of a static residual in $\Delta\alpha$ can be formalized by splitting the perturbed angle-of-attack variable into a fast component independent of the slow mode dynamics, and a slow component that follows the slow mode dynamics, that is, $\Delta\alpha = \Delta\alpha_f + \Delta\alpha_s$. With this splitting incorporated, Eqs. (3) and (4) can be written as

$$\ddot{\Delta}\alpha_f - \left[\left(\frac{g}{V_0}\right)z_u\right]\dot{\Delta}v - \left[\sqrt{\frac{\bar{q}Sc}{I_y}}m_q + \left(\frac{g}{V_0}\right)z_\alpha\right]\dot{\Delta}\alpha_f - \left[\frac{\bar{q}Sc}{I_y}m_u - \sqrt{\frac{\bar{q}Sc}{I_y}}\left(\frac{g}{V_0}\right)m_q z_u\right]\Delta v - \left[\frac{\bar{q}Sc}{I_y}m_\alpha - \sqrt{\frac{\bar{q}Sc}{I_y}}\left(\frac{g}{V_0}\right)m_q z_\alpha\right]\Delta\alpha_f + \ddot{\Delta}\alpha_s - \left[\sqrt{\frac{\bar{q}Sc}{I_y}}m_q + \left(\frac{g}{V_0}\right)z_\alpha\right]\dot{\Delta}\alpha_s - \left[\frac{\bar{q}Sc}{I_y}m_\alpha - \sqrt{\frac{\bar{q}Sc}{I_y}}\left(\frac{g}{V_0}\right)m_q z_\alpha\right]\Delta\alpha_s = 0 \tag{7}$$

$$\ddot{\Delta}v - \left[\left(\frac{g}{V_0}\right)x_u\right]\dot{\Delta}v - \left[\left(\frac{g}{V_0}\right)(x_\alpha - 1)\right]\dot{\Delta}\alpha_f - \left[\left(\frac{g}{V_0}\right)^2 z_u\right]\Delta v - \left[\left(\frac{g}{V_0}\right)^2 z_\alpha\right]\Delta\alpha_f - \left[\left(\frac{g}{V_0}\right)(x_\alpha - 1)\right]\dot{\Delta}\alpha_s - \left[\left(\frac{g}{V_0}\right)^2 z_\alpha\right]\Delta\alpha_s = 0 \tag{8}$$

where it is understood that Δv varies as per the slow timescale, and hence no subscript is used on Δv. Rescaling the time derivatives of the fast variable $\Delta\alpha_f$ with t_f (denoted by $'$) and the time derivatives of the slow variables $\Delta\alpha_s$ and Δv with t_s (denoted by *), in Eqs. (7) and (8), gives the following equations for the fast and slow modes, respectively:

$$\Delta\alpha_f'' - (m_q + \epsilon z_\alpha)\Delta\alpha_f' - (m_\alpha - \epsilon m_q z_\alpha)\Delta\alpha_f + \epsilon^2\Delta\alpha_s^{**} - \epsilon^2 z_u\Delta v^* - \epsilon(m_q + \epsilon z_\alpha)\Delta\alpha_s^* - (m_u - \epsilon m_q z_u)\Delta v - (m_\alpha - \epsilon m_q z_\alpha)\Delta\alpha_s = 0 \tag{9}$$

$$\Delta v^{**} - x_u\Delta v^* - z_u\Delta v - (x_\alpha - 1)\Delta\alpha_s^* - z_\alpha\Delta\alpha_s - 1/\epsilon(x_\alpha - 1)\Delta\alpha_f' - z_\alpha\Delta\alpha_f = 0 \tag{10}$$

where ϵ is the small parameter defined in Eq. (5). With this rescaling all variables and their derivatives in Eqs. (9) and (10) are dimensionless and of the same order, and the order of each term is now just the order of its coefficient in ϵ. The longitudinal dynamics in the form of Eqs. (9) and (10) is now suitable for a consistent order-by-order analysis.

III. Zeroth-Order Approximation

As shown in Ref. 6, the relation between the slow component of the angle of attack $\Delta\alpha_s$ and the slow mode variable Δv is of the form

$$\Delta\alpha_s = K_0\Delta v \tag{11}$$

where K_0 is yet to be determined. Using Eq. (11) to replace $\Delta\alpha_s$ and its derivatives in Eq. (9) and dropping all terms of order one or higher in ϵ, gives the following equation for the fast mode in the zeroth-order approximation:

$$\Delta\alpha_f'' - m_q\Delta\alpha_f' - m_\alpha\Delta\alpha_f = (m_u + K_0 m_\alpha)\Delta v \tag{12}$$

where the left-hand side is the short period dynamics in the fast component $\Delta\alpha_f$. However, because $\Delta\alpha_f$ is, by definition, the component of $\Delta\alpha$ that is not influenced by the slow mode dynamics in Δv, the coefficient on the right-hand side of Eq. (12) must be equal to zero. This gives $K_0 = -m_u/m_\alpha$, whereas the nondimensional short period frequency and damping to zeroth order can be read from the left-hand side of Eq. (12) to be

$$\hat{\omega}_{SP}^2 = -m_\alpha, \qquad 2\hat{\omega}_{SP}\hat{\zeta}_{SP} = -m_q \tag{13}$$

(The caret symbol indicates short period parameters nondimensionalized with the fast timescale T_f.) The steady-state value of $\Delta\alpha_f$, from Eq. (12), is clearly zero. Hence, putting $\Delta\alpha_f$ and its derivatives to zero in the slow mode Eq. (10) and using Eq. (11) for $\Delta\alpha_s$, the zeroth-order approximation to the phugoid dynamics is obtained as

$$\Delta v^{**} - [x_u + K_0(x_\alpha - 1)]\Delta v^* - (z_u + K_0 z_\alpha)\Delta v = 0 \tag{14}$$

Substituting for K_0 from the preceding, Eq. (14) gives the zeroth-order approximation to the nondimensional phugoid frequency and damping to be

$$\tilde{\omega}_P^2 = -[z_u - (m_u/m_\alpha)z_\alpha]$$
$$2\tilde{\omega}_P\tilde{\zeta}_P = -[x_u - m_u/m_\alpha(x_\alpha - 1)] \tag{15}$$

(The tilde symbol indicates phugoid parameters nondimensionalized with the slow timescale T_s.) When nondimensionalized with the fast timescale T_f, the phugoid approximation in Eq. (15) can be written as

$$\hat{\omega}_P^2 = -\epsilon^2[z_u - (m_u/m_\alpha)z_\alpha]$$
$$2\hat{\omega}_P\hat{\zeta}_P = -\epsilon[x_u - m_u/m_\alpha(x_\alpha - 1)] \tag{16}$$

where the factors of ϵ indicate that the phugoid mode is an order slower than the short period mode. On replacing the nondimensional terms in Eqs. (13) and (16) with the corresponding dimensional derivatives, we get the zeroth order approximation to the short period and phugoid modes next:

$$\omega_{\rm SP}^2 = [-M_\alpha]_0, \qquad 2\omega_{\rm SP}\zeta_{\rm SP} = [-M_q]_0$$
$$\omega_P^2 = \left[\left(\frac{g}{V_0}\right)\left(-Z_u + \frac{M_u}{M_\alpha}Z_\alpha\right)\right]_2$$
$$2\omega_P\zeta_P = \left[-X_u + \frac{M_u}{M_\alpha}(X_\alpha - g)\right]_1 \tag{17}$$

The subscripts on the preceding right-hand-side expressions indicate the order in ϵ of the respective bracketed terms when nondimensionalized by the fast timescale T_f.

The consistent zeroth-order approximation in Eq. (17) will now be used to calculate the coefficients of the characteristic polynomial, which will then be compared with the coefficients of the exact characteristic polynomial listed in the Appendix. It is important to realize that each coefficient of the characteristic polynomial computed from a consistent literal approximation will itself be consistent only up to a particular order. To see this, consider the coefficient B in Eq. (A3) from the Appendix. Of the two contributors to coefficient B, the zeroth-order approximation in Eq. (17) gives $2\zeta_{\rm SP}\omega_{\rm SP}$ to order zero, whereas $2\zeta_P\omega_P$, by virtue of being a slow mode parameter, has no order zero term but has a leading term of order one. At this level of approximation, however, order one terms are not available for the fast mode parameter $2\zeta_{\rm SP}\omega_{\rm SP}$. They can be captured only by a first-order approximation, which is carried out in the next section. Meanwhile, the coefficient B computed from the zeroth-order approximation can be compared with the exact expression in the Appendix only to zeroth-order. Defining the ϵ order of a quantity to be the highest order to which all terms in that quantity are consistently available, the ϵ order for coefficient B computed from the zeroth-order approximation is zero. The ϵ order for the other coefficients calculated from the zeroth-order approximation can be similarly determined. The coefficients of the characteristic polynomial, as defined in the Appendix, computed from the zeroth-order approximation are shown next, where the subscripts on the coefficients indicate the correct ϵ order:

$$B_0 = [2\zeta_{\rm SP}\omega_{\rm SP} + 2\zeta_P\omega_P]_0 = [-M_q]_0$$
$$C_0 = \left[\omega_{\rm SP}^2 + \omega_P^2 + (2\zeta_{\rm SP}\omega_{\rm SP})(2\zeta_P\omega_P)\right]_0 = [-M_\alpha]_0$$
$$D_1 = \left[(2\zeta_{\rm SP}\omega_{\rm SP})\omega_P^2 + (2\zeta_P\omega_P)\omega_{\rm SP}^2\right]_1 = [X_u M_\alpha - (X_\alpha - g)M_u]_1$$
$$E_2 = \left[\omega_{\rm SP}^2\omega_P^2\right]_2 = [(g/V_0)(M_\alpha Z_u - M_u Z_\alpha)]_2$$

Comparison with Eqs. (A3–A6) in the Appendix shows a perfect match to the respective ϵ orders, which confirms that there are no missing terms at the leading order in the approximations in Eq. (17).

It is interesting to compare the zeroth-order phugoid approximation in Eq. (17) with the standard phugoid approximation in the literature (for example, see Ref. 4, p. 6.34). The standard approximations, which are given by $\omega_P^2 = -gZ_u/V_0$ and $2\omega_P\zeta_P = -X_u$, can each be seen to have a missing term at the leading order, which explains why they are usually found to be unsatisfactory. The standard short period approximation (for example, see Ref. 4, pp. 6.30, 6.31) is given by $\omega_{\rm SP}^2 = -M_\alpha + M_q(Z_\alpha/V_0)$ and $2\omega_{\rm SP}\zeta_{\rm SP} = -M_q - Z_\alpha/V_0$. The first term in each of these expressions is identical to the zeroth-order approximation in Eq. (17), whereas the second term must clearly be of higher order because all zeroth-order terms have been consistently accounted for in the approximations in Eq. (17). The differing orders of the two terms in the existing short period frequency and damping approximations appear to have been overlooked so far.

IV. First-Order Approximation

The approach followed here is the same as that for the zeroth-order approximation in the preceding section, and the presentation is along similar lines. A more general version of Eq. (11) needs to be considered here, to account for the higher-order effects, as follows:

$$\Delta\alpha_s = K_0\Delta v + \epsilon(K_1\Delta v + J_1\Delta v^*) \tag{18}$$

where K_0 is the same as that obtained in the zeroth-order approximation. [No term of the form $J_0\Delta v^*$ is used in Eq. (11) or (18) as it can be shown from the zeroth-order approximation that $J_0 = 0$.] Using Eq. (18) with Eq. (9), and now keeping all terms of order zero and one in ϵ, gives the following first-order approximation to the fast mode dynamics:

$$[1 + \epsilon J_1(x_\alpha - 1)]\Delta\alpha_f'' - [m_q + \epsilon z_\alpha - \epsilon(x_\alpha - 1)(K_0 - m_q J_1)]\Delta\alpha_f'$$
$$- (m_\alpha - \epsilon m_q z_\alpha)\Delta\alpha_f = \epsilon(m_q K_0 + m_\alpha J_1)\Delta v^*$$
$$+ [m_u - \epsilon m_q z_u + (m_\alpha - \epsilon m_q z_\alpha)K_0 + \epsilon m_\alpha K_1]\Delta v \tag{19}$$

As before, the coefficients of Δv and Δv^* on the right-hand side must each be zero, which allows one to solve for K_1 and J_1 as follows:

$$K_1 = m_q/m_\alpha[z_u - (m_u/m_\alpha)z_\alpha], \qquad J_1 = m_q/m_\alpha(m_u/m_\alpha) \tag{20}$$

Inserting the solutions for K_1 and J_1 from Eq. (20) into the left-hand side of Eq. (19) gives the first-order approximation to the nondimensional short period frequency and damping as

$$\hat{\omega}_{\rm SP}^2 = -m_\alpha + \epsilon m_q[z_\alpha + m_u/m_\alpha(x_\alpha - 1)]$$
$$2\hat{\omega}_{\rm SP}\hat{\zeta}_{\rm SP} = -m_q - \epsilon[z_\alpha + m_u/m_\alpha(x_\alpha - 1)] \tag{21}$$

where the terms of order zero in ϵ are expectedly the same as those from the zeroth-order approximation in Eq. (13), and additional terms of order ϵ have now been captured. Then, using Eq. (18) in Eq. (10) gives the following first-order approximation to the phugoid dynamics:

$$\Delta v^{**} - (x_u + K_0(x_\alpha - 1) + \epsilon\{z_\alpha J_1 + (x_\alpha - 1)[K_1 + x_u J_1$$
$$+ (x_\alpha - 1)K_0 J_1]\})\Delta v^* - \{z_u + K_0 z_\alpha + \epsilon[z_\alpha K_1$$
$$+ (x_\alpha - 1)(z_u + z_\alpha K_0)J_1]\}\Delta v = 0 \tag{22}$$

which, on substituting for K_0 from the preceding section and for K_1 and J_1 from Eq. (20), gives the first-order approximation to the nondimensional phugoid frequency and damping as follows:

$$\tilde{\omega}_P^2 = -\left(z_u - \frac{m_u}{m_\alpha}z_\alpha\right) - \epsilon\frac{m_q}{m_\alpha}\left(z_u - \frac{m_u}{m_\alpha}z_\alpha\right)\left[z_\alpha + \frac{m_u}{m_\alpha}(x_\alpha - 1)\right]$$

$$2\tilde{\omega}_P\tilde{\zeta}_P = -\left[x_u - \frac{m_u}{m_\alpha}(x_\alpha - 1)\right] - \epsilon\frac{m_q}{m_\alpha}$$
$$\times\left(\frac{m_u}{m_\alpha}z_\alpha + (x_\alpha - 1)\left\{\left(z_u - \frac{m_u}{m_\alpha}z_\alpha\right)\right.\right.$$
$$\left.\left.+ \frac{m_u}{m_\alpha}\left[x_u - \frac{m_u}{m_\alpha}(x_\alpha - 1)\right]\right\}\right) \tag{23}$$

Once again, terms of order zero in ϵ appearing in Eq. (23) are confirmed to be identical to those that occur in the zeroth-order approximation in Eq. (15). When nondimensionalized with respect to the fast timescale T_f, the phugoid parameters in Eq. (23) appear as

$$\hat{\omega}_P^2 = -\epsilon^2\left(z_u - \frac{m_u}{m_\alpha}z_\alpha\right) - \epsilon^3\frac{m_q}{m_\alpha}\left(z_u - \frac{m_u}{m_\alpha}z_\alpha\right) \times \left[z_\alpha + \frac{m_u}{m_\alpha}(x_\alpha - 1)\right]$$

$$2\hat{\omega}_P\hat{\zeta}_P = -\epsilon\left[x_u - \frac{m_u}{m_\alpha}(x_\alpha - 1)\right] - \epsilon^2\frac{m_q}{m_\alpha} \times \left(\frac{m_u}{m_\alpha}z_\alpha + (x_\alpha - 1)\left\{\left(z_u - \frac{m_u}{m_\alpha}z_\alpha\right) + \frac{m_u}{m_\alpha}\left[x_u - \frac{m_u}{m_\alpha}(x_\alpha - 1)\right]\right\}\right) \tag{24}$$

Replacing the nondimensional bookkeeping terms in Eqs. (21) and (24) by their corresponding dimensional derivatives gives the following first-order approximation to the short period and phugoid modes, where the subscripts indicate the order in ϵ of each bracketed expression when made nondimensional with the fast timescale T_f:

$$\omega_{\text{SP}}^2 = [-M_\alpha]_0 + \left[M_q\left(\frac{Z_\alpha}{V_0}\right)\right]_1 + \left[M_q\left(\frac{M_u}{M_\alpha}\right)(X_\alpha - g)\right]_1$$

$$2\omega_{\text{SP}}\zeta_{\text{SP}} = [-M_q]_0 + \left[-\frac{Z_\alpha}{V_0}\right]_1 + \left[-\frac{M_u}{M_\alpha}(X_\alpha - g)\right]_1$$

$$\omega_P^2 = \left[-\frac{g}{V_0}\left(Z_u - \frac{M_u}{M_\alpha}Z_\alpha\right)\right]_2 + \left\{-\frac{g}{V_0}\left(Z_u - \frac{M_u}{M_\alpha}Z_\alpha\right)\frac{M_q}{M_\alpha}\left[\frac{Z_\alpha}{V_0} + \frac{M_u}{M_\alpha}(X_\alpha - g)\right]\right\}_3$$

$$2\omega_P\zeta_P = \left[-X_u + \frac{M_u}{M_\alpha}(X_\alpha - g)\right]_1 + \left[-\frac{M_q}{M_\alpha}\frac{M_u}{M_\alpha}\left(\frac{gZ_\alpha}{V_0}\right)\right]_2 + \left[-(X_\alpha - g)\frac{M_q}{M_\alpha}\frac{1}{V_0}\left(Z_u - \frac{M_u}{M_\alpha}Z_\alpha\right)\right]_2 + \left\{(X_\alpha - g)\frac{M_q}{M_\alpha}\frac{M_u}{M_\alpha}\left[-X_u + \frac{M_u}{M_\alpha}(X_\alpha - g)\right]\right\}_2 \tag{25}$$

The preceding first-order approximations are used to calculate the coefficients of the characteristic polynomial to their respective ϵ orders, as shown next:

$$B_1 = [2\zeta_{\text{SP}}\omega_{\text{SP}} + 2\zeta_P\omega_P]_1 = [-M_q]_0 + [-X_u - Z_\alpha/V_0]_1$$

$$C_1 = \left[\omega_{\text{SP}}^2 + \omega_P^2 + (2\zeta_{\text{SP}}\omega_{\text{SP}})(2\zeta_P\omega_P)\right]_1 = [-M_\alpha]_0 + [M_q(X_u + Z_\alpha/V_0)]_1$$

$$D_2 = \left[(2\zeta_{\text{SP}}\omega_{\text{SP}})\omega_P^2 + (2\zeta_P\omega_P)\omega_{\text{SP}}^2\right]_2 = [X_uM_\alpha - (X_\alpha - g)M_u]_1 + \{M_q[(X_\alpha Z_u - X_uZ_\alpha)/V_0]\}_2$$

$$E_3 = \left[\omega_{\text{SP}}^2\omega_P^2\right]_3 = [(g/V_0)(M_\alpha Z_u - M_uZ_\alpha)]_2$$

The computed coefficients can be seen to match perfectly the corresponding expressions in Eqs. (A3–A6) in the Appendix to the correct orders. (Note that the order three term in the preceding coefficient E is correctly found to be zero.)

The consistent first-order short period frequency and damping approximations in Eq. (25) are made up of three terms each. In contrast, the standard short period approximation cited from Ref. 4 in the preceding section can be seen to contain only the first two of these three terms. The third term, discovered in this derivation, is caused by the influence of the slow mode variables on the short period dynamics and happens to be of the same order as the second term. The existing short period approximation is, therefore, not consistent at the first order. We have already shown in the preceding section that the standard phugoid approximation is not consistent even at the zeroth order. Later derivations for the phugoid frequency[5,6] were consistent to zeroth order, but not to the first order. The consistent first-order phugoid frequency approximation in Eq. (25) can be rewritten, after some algebraic manipulation, as

$$\omega_P^2 = \frac{g(M_uZ_\alpha - M_\alpha Z_u)}{M_\alpha V_0 - M_qZ_\alpha - M_qM_uV_0(X_\alpha - g)} \tag{26}$$

and is seen to contain an additional third term in the denominator that has not been recognized in earlier derivations.[5] In case of phugoid damping, several terms in the consistent first-order approximation in Eq. (25) can be seen to be missing from previous derivations for the phugoid damping.[5]

V. Conclusions

Consistent zeroth- and first-order approximations to the longitudinal modes have been derived and are shown to correctly compute all of the coefficients of the characteristic polynomial to the appropriate order. This is a significant achievement as previously derived approximations have never succeeded in satisfying the characteristic polynomial. The present derivation has ensured that there are no missing terms in the consistent approximations at each order. In this sense the frequency and damping expressions reported in this Note are the best possible at each order. Corrections to the existing literal approximations for the short period and phugoid modes are discovered. Finally, it might be appropriate to mention that the approximations derived in this Note are not specific to any select set of aircraft data and should be viewed as results of a fundamental nature. In particular, questions regarding the relative numerical significance of the different terms in any of the expressions derived here are not of relevance.

Appendix: Longitudinal Characteristic Polynomial

The longitudinal dynamics given by Eqs. (1) and (2), being a set of two second-order differential equations, have an associated fourth-order characteristic polynomial $\boldsymbol{P}_{\text{lon}}$ that factors as follows:

$$\boldsymbol{P}_{\text{lon}}(\lambda) = \left(\lambda^2 + 2\zeta_{\text{SP}}\omega_{\text{SP}}\lambda + \omega_{\text{SP}}^2\right)\left(\lambda^2 + 2\zeta_P\omega_P\lambda + \omega_P^2\right) \tag{A1}$$

where ω and ζ are the exact frequency and damping parameters. Let the coefficients of $\boldsymbol{P}_{\text{lon}}$ be labeled from A through E, as follows:

$$\boldsymbol{P}_{\text{lon}}(\lambda) = A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E \tag{A2}$$

where the exact expressions for the coefficients A through E are available (for example, see Ref. 4, p. 6.20). Then, equating coefficients of like powers in λ between Eqs. (A1) and (A2) gives the following relations to be satisfied by the frequency and damping parameters, with $A = 1$:

$$B = 2\zeta_{\text{SP}}\omega_{\text{SP}} + 2\zeta_P\omega_P = [-M_q]_0 + [-X_u - Z_\alpha/V_0]_1 \tag{A3}$$

$$C = \omega_{\text{SP}}^2 + \omega_P^2 + (2\zeta_{\text{SP}}\omega_{\text{SP}})(2\zeta_P\omega_P) = [-M_\alpha]_0 + [M_q(X_u + Z_\alpha/V_0)]_1 + [(X_uZ_\alpha - X_\alpha Z_u)/V_0]_2 \tag{A4}$$

$$D = (2\zeta_{\text{SP}}\omega_{\text{SP}})\omega_P^2 + (2\zeta_P\omega_P)\omega_{\text{SP}}^2 = [X_uM_\alpha - (X_\alpha - g)M_u]_1 + [M_q(X_\alpha Z_u - X_uZ_\alpha)/V_0]_2 \tag{A5}$$

$$E = \omega_{\text{SP}}^2\omega_P^2 = [(g/V_0)(M_\alpha Z_u - M_uZ_\alpha)]_2 \tag{A6}$$

The subscripts on the square brackets indicate the order in ϵ of the terms contained in the bracket when nondimensionalized by the fast timescale T_f.

References

[1]Lanchester, F. W., *Aerodonetics*, Archibald Constable, London, 1908.

[2]Etkin, B., and Reid, L. D., *Dynamics of Flight: Stability and Control*, Wiley, New York, 1996.

[3]Nelson, R. C., *Flight Stability and Automatic Control*, McGraw–Hill, New York, 1990.

[4]Roskam, J., *Airplane Flight Dynamics and Automatic Flight Controls*, Vol. 1, Roskam Aviation and Engineering Corp., Lawrence, KS, 1979.

[5]Pradeep, S., "A Century of Phugoid Approximations," *Aircraft Design*, Vol. 1, 1998, pp. 89–104.

[6]Ananthkrishnan, N., and Unnikrishnan, S., "Literal Approximations to Aircraft Dynamic Modes," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 6, 2001, pp. 1196–1203.

[7]Phillips, W. F., "Phugoid Approximation for Conventional Airplanes," *Journal of Aircraft*, Vol. 37, No. 1, 2000, pp. 30–36.

Rigid-Body Trajectory Reconstruction

R. Scott McEntire*
Sandia National Laboratories, Albuquerque, New Mexico 87185
Steven L. Folkman†
Utah State University, Logan, Utah 84322-4130
and
Ned R. Hansen‡
Sandia National Laboratories, Albuquerque, New Mexico 87185

Introduction

It is desired to reconstruct the path of a moving body as it impacts a target at a high velocity using only linear accelerometers as sensors. The path of the penetrator is described by three translations and three rotations. When kinematic relationships are used for a rigid body, a system of six ordinary differential equations can be established relating the six degrees of freedom to accelerations. A minimum of six acceleration measurements are required to solve for the six degrees of freedom at each time step. However, using six accelerometers can lead to errors with divergent (unstable) solutions. If more than six accelerometers are used, the system of equations is overdetermined. This extra information provides a stabilizing influence on the solution. Schuler et al.[1] proved the instability of one specific six-accelerometer configuration. Padgaonkar et al.[2] discussed using a six-accelerometer approach and noted that acceleration data must be accurate to approximately 1% of peak values. Padgaonkar et al. proposed a nine-accelerometer solution be used when the instrumented body experiences a direct impact with a hard surface or when low-sensitivity accelerometers are used. Mital and King[3] also proposed a nine-accelerometer configuration and determined their method to be very accurate. The location and orientation of the accelerometers is very important to the solution of the problem. However, a procedure for selecting the optimum number of transducers, their location, and their orientation was not found during a review of previous studies.

Solution Procedures

The penetrator is illustrated in Fig. 1 by the dashed box. Point O is fixed in a Newtonian reference frame, and B^* is the mass center of a rigid-body penetrator. We will use unit vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$ fixed at O and $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$ fixed in B. Six generalized coordinates (q_1–q_6) describe the position and orientation of a rigid body. The position vector of B^* be given by $\boldsymbol{r}^{B^*}$ or

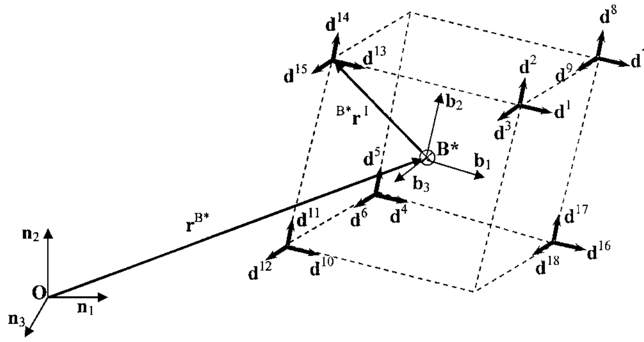

Fig. 1 Coordinate system of rigid-body penetrator and location of accelerometers $\boldsymbol{d}^i$.

$$\boldsymbol{r}^{B^*} \equiv q_1\boldsymbol{n}_1 + q_2\boldsymbol{n}_2 + q_3\boldsymbol{n}_3 \tag{1}$$

The orientation of B in reference frame N is specified by three generalized coordinates, q_4, q_5, and q_6, which are successive rotations. A body-three 3-2-1 rotation sequence is assumed: a rotation q_4 about $\boldsymbol{b}_3$, then q_5 about $\boldsymbol{b}_2$, and finally q_6 about $\boldsymbol{b}_1$. We define six generalized speeds, u_1, u_2, u_3, u_4, u_5, and u_6, such that the velocity and angular velocity of the mass center are given by

$${}^N\boldsymbol{v}^{B^*} \equiv u_1\boldsymbol{n}_1 + u_2\boldsymbol{n}_2 + u_3\boldsymbol{n}_3, \qquad {}^N\boldsymbol{\omega}^B \equiv u_4\boldsymbol{b}_1 + u_5\boldsymbol{b}_2 + u_6\boldsymbol{b}_3 \tag{2}$$

This study limited the transducers to 18 locations on the corners of a unit cube, as shown in Fig. 1. The length dimension of the cube is 1 m, and all accelerations are reported in meters per second squared. However, one could also consider the cube dimension to be 1 ft and all accelerations to be in feet per second squared.

It can be shown that the equations of motion of the projectile can be described as

$$\left[H_j^I\right]\{\dot{u}_j\} = \{w^I\}, \qquad I = 1,\ldots,N_T \qquad j = 1,\ldots,6 \tag{3}$$

where N_T is the number of transducers and

$$H_1^I \equiv \delta_1^I C_{11} + \delta_2^I C_{12} + \delta_3^I C_{13}, \qquad H_2^I \equiv \delta_1^I C_{21} + \delta_2^I C_{22} + \delta_3^I C_{23}$$

$$H_3^I \equiv \delta_1^I C_{31} + \delta_2^I C_{32} + \delta_3^I C_{33}, \qquad H_4^I \equiv \delta_3^I \rho_2^I - \delta_2^I \rho_3^I$$

$$H_5^I \equiv \delta_1^I \rho_3^I - \delta_3^I \rho_1^I, \qquad H_6^I \equiv \delta_2^I \rho_1^I - \delta_1^I \rho_2^I$$

$$w^I \equiv M^I - \left(\delta_1^I Z_4^I + \delta_2^I Z_5^I + \delta_3^I Z_6^I\right)$$

$$Z_4^I = u_4u_5\rho_2^I + u_4u_6\rho_3^I - \left(u_5^2 + u_6^3\right)\rho_1^I$$

$$Z_5^I = u_4u_5\rho_1^I + u_5u_6\rho_3^I - \left(u_4^2 + u_6^3\right)\rho_2^I$$

$$Z_6^I = u_4u_6\rho_1^I + u_5u_6\rho_2^I - \left(u_4^2 + u_5^3\right)\rho_3^I, \qquad I = 1,\ldots,N_T \tag{4}$$

The ρ variable is used to define the location of transducer I relative to the mass center as

$${}^{B^*}\boldsymbol{r}^I \equiv \rho_1^I\boldsymbol{b}_1 + \rho_2^I\boldsymbol{b}_2 + \rho_3^I\boldsymbol{b}_3, \qquad I = 1,\ldots,N_T \tag{5}$$

and a unit vector defining the sensing direction of transducer I is

$$\boldsymbol{d}^I \equiv \delta_1^I\boldsymbol{b}_1 + \delta_2^I\boldsymbol{b}_2 + \delta_3^I\boldsymbol{b}_3, \qquad I = 1,\ldots,N_T \tag{6}$$

The body rotation matrices C_{ij} are defined as[4]

Received 30 August 2001; revision received 7 March 2002; accepted for publication 12 March 2002.

*Technical Staff, Firing Set and Optical Engineering Department.

†Professor, Department of Mechanical and Aerospace Engineering. Senior Member AIAA.

‡Technical Staff, Advanced Systems Department; currently Chief Engineer, Engineering Department, S&S Power, Inc., Logan, UT 84321.